SQUEEZED STATES AS REPRESENTATIONS OF SYMPLECTIC GROUPS

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Abstract

It is shown that the $SU(1,1)$-like and $SU(2)$-like two-photon coherent states can be combined to form a $O(3,2)$-like two-photon states. Since the $O(3,2)$ group has many subgroups, there are also many new interesting new coherent and squeezed two-photon states. Among them is the two-photon sheared state whose symmetry property is like that for the two-dimensional Euclidean group. There are now two-phonon coherent states which may exhibit symmetries not yet observed for photons, including sheared states. Let us note that both $SU(1,1)$ and $S(3,2)$ are isomorphic to the symplectic groups $Sp(2)$ and $Sp(4)$ respectively, and that symplectic transformations consist of rotations and squeeze transformations.

I. INTRODUCTION

We are quite familiar with rotations, but not with squeeze transformations. Yet, squeeze transformations are everywhere in physics, including special relativity, classical mechanics, and quantum mechanics. If we enlarge the group of rotations by including squeeze transformations, it becomes a symplectic group. For this reason, most of the groups in physics are symplectic groups or subgroups of the symplectic group, and quantum optics is not an exception. We have squeezed states of light!

Let us first look at the two two-mode states discussed in detail by Yurke, McCall and Klauder [1]. One of the states is generated by the operators satisfying the commutation relations for the $SU(2)$ group, and the other state generated by those satisfying the $SU(1,1)$ algebra or the algebra of the $Sp(2)$ symplectic group [2,3]. If we combine these two states by constructing a set of closed commutation relations starting from the generators of both

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states, we end up with ten generators satisfying the commutation relations for the $O(3,2)$ Lorentz group [3–5]. This $O(3,2)$-like two-oscillator state was considered by Dirac in 1963 [6].

Although quantum optics has been the primary beneficiary of the symmetries exhibited by two-photon states in recent years, we should note that squeezed states are not restricted to optical sciences. It was noted that the $O(3,2)$ symmetry is relevant to Bogoliubov transformations in superconductivity [7]. There are now two-phonon coherent states [8,9] possessing symmetry properties not observed in photon states. It is quite possible that they symmetries derivable from the $E(2)$-like subgroups of the $O(3,2)$-like two-oscillator system.

In Secs II, III, and IV we discuss how the $O(3,2)$ states are constructed from the $SU(2)$ and $SU(1,1)$ states, the $E(2)$-like little group of $O(3,2)$, and the $E(2)$-like sheared states, respectively. Our discussion of the sheared state is largely based on the 1992 paper by Kim and Yeh [10].

II. SQUEEZED STATES

It is a well-established tradition that coherent and squeezed states as well as their variants are represented by three-parameter groups. The symmetry of coherent state is a representation of the three-parameter Heisenberg group.

The one-mode squeezed state is representation of the $U(1,1)$ group generated by

$$\frac{1}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right), \qquad \frac{1}{2}\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a}\right), \qquad \frac{i}{2}\left(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a}\right). \tag{2.1}$$

which is locally isomorphic to $Sp(2)$, generated by

$$\frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{2.2}$$

which satisfy the same set of commutation relations as that for the generators of the $SU(1,1)$ group.

For two-mode squeezed states, the $SU(2)$ and $SU(1,1)$ symmetries have been discussed extensively in the literature [1,3,11]. Those are generated respectively

$$\hat{J}_1 = \frac{1}{2}\left(\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2\right), \quad \hat{J}_2 = \frac{1}{2}\left(\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1\right), \quad \hat{J}_3 = \frac{1}{2i}\left(\hat{a}_1^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_1\right) \tag{2.3}$$

and

$$\hat{J}_o = \frac{1}{2}\left(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2\hat{a}_2^\dagger\right), \quad \hat{K}_1 = \frac{1}{2}\left(\hat{a}_1^\dagger\hat{a}_2^\dagger + \hat{a}_1\hat{a}_2\right), \quad \hat{Q}_1 = \frac{i}{2}\left(\hat{a}_1^\dagger\hat{a}_2^\dagger - \hat{a}_1\hat{a}_2\right). \tag{2.4}$$

If we take commutation relations of the six operators given in Eq.(2.3) and Eq.(2.4), they generate the following four additional generators.

$$\hat{K}_2 = -\frac{1}{4}\left(\hat{a}_1^\dagger\hat{a}_1^\dagger + \hat{a}_1\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2^\dagger - \hat{a}_2\hat{a}_2\right), \quad \hat{K}_3 = \frac{i}{4}\left(\hat{a}_1^\dagger\hat{a}_1^\dagger - \hat{a}_1\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2^\dagger - \hat{a}_2\hat{a}_2\right),$$

$$\hat{Q}_2 = -\frac{i}{4}\left(\hat{a}_1^\dagger\hat{a}_1^\dagger - \hat{a}_1\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2^\dagger + \hat{a}_2\hat{a}_2\right), \quad \hat{Q}_3 = -\frac{1}{4}\left(\hat{a}_1^\dagger\hat{a}_1^\dagger + \hat{a}_1\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2^\dagger + \hat{a}_2\hat{a}_2\right). \tag{2.5}$$

There are ten linearly independent generators. In terms of the Pauli matrices, they are [12]

$$J_1 = \frac{i}{2}\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}, \quad J_2 = \frac{i}{2}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \quad J_3 = \frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad J_0 = \frac{i}{2}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

$$K_1 = \frac{-i}{2}\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad K_2 = \frac{i}{2}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad K_3 = \frac{i}{2}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

$$Q_1 = \frac{i}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \quad Q_2 = \frac{-i}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad Q_3 = \frac{i}{2}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{2.6}$$

These generators satisfy the commutation relations:

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k, \quad [J_i, Q_j] = i\epsilon_{ijk}Q_k, \quad [K_i, Q_j] = i\delta_{ij}J_0,$$

$$[K_i, K_j] = [Q_i, Q_j] = -i\epsilon_{ijk}J_k, \quad [J_i, J_o] = 0, \quad [K_i, J_0] = iQ_i, \quad [Q_i, J_0] = -iK_i. \tag{2.7}$$

Indeed, these are the generators of the group $Sp(4)$ which is locally isomorphic to the $(3 + 2)$-dimensional deSitter group [5,6].

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III. THREE-PARAMETER SUBGROUPS OF SP(4)

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The group $Sp(4)$ has many subgroups, and three-parameter subgroups are particularly useful in studying two-mode squeezed states [1,5]. The most obvious subgroup is the $SU(2)$-like group generated by J_1, J_2, and J_3. Another subgroup frequently discussed in the literature is the $SU(1,1)$-like group generated by J_o, K_3, Q_3. From the local isomorphism between $Sp(4)$ and $O(3,2)$, it is not difficult to find all possible $SU(1,1)$-like subgroups.

Indeed, the above-mentioned $SU(1,1)$-like subgroup is unitarily equivalent to those generated by J_o, K_1, Q_1 and by J_0, K_2, Q_2. There is also an $SU(1,1)$-like subgroup generated by K_1, K_2, J_3, which is unitarily equivalent to Q_1, Q_2, and J_3. These are then unitarily equivalent to the subgroups generated by K_2, K_3, J_1, by K_3, K_1, J_2, by Q_2, Q_3, J_1, and by Q_3, Q_1, J_2. It is known from the Lorentz group that the signs of K_i and Q_i can be changed.

The purpose of this paper is to discuss additional three-parameter subgroups. It is known that the $O(3,1)$ Lorentz group has a number of $E(2)$-like subgroups. Since there are two $O(3,1)$-like subgroups in the $O(3,2)$ deSitter group, the group $Sp(4)$ should contain a number of $E(2)$-like subgroups [12]. If we define

$$F_1 = K_1 - J_2, \qquad F_2 = K_2 + J_1, \tag{3.1}$$

and

$$G_1 = Q_1 - J_2, \qquad G_2 = Q_2 + J_1, \tag{3.2}$$

then the resulting commutation relations are

$$[F_1, F_2] = 0, \qquad [J_3, F_1] = -iF_2, \qquad [J_3, F_2] = iF_1, \tag{3.3}$$

and similar relations for G_1, G_2 and J_3. This set of commutation relations is the same as for the generators of the $E(2)$ group. The subgroup generated by G_1, G_2 and J_3 is unitarily equivalent to that generated by F_1, F_2 and J_3. There are four-additional subgroups generated by F_2, F_3, J_1, by F_3, F_1, J_2, by G_2, G_3, J_1, and by G_3, G_1, J_2. They are all unitarily equivalent. Thus we can study them all by studying one of them.

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IV. CONSTRUCTION OF SHEARED STATES

Let us define the word "shear." In the xp plane, we can consider linear transformations of the type

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}, \tag{4.1}$$

Under this transformation, the x coordinate undergoes a translation proportional to p while the p variable remains unchanged.

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}. \tag{4.2}$$

For a given function $f(x, p)$, the shears of Eq.(4.1) and Eq.(4.2) lead to $f(x - \alpha p, p)$ and $f(x, p - \alpha x)$ respectively.

With this preparation, we continue our discussion of the $E(2)$-like subgroups discussed introduced in Sec. IV. In the four-by-four matrix representation, the rotation generator J_3 is given in Eq.(2.6). The generators F_1 and F_2 take the matrix form.

$$F_1 = -i \begin{pmatrix} 0 & \sigma_1 \\ 0 & 0 \end{pmatrix}, \qquad F_2 = i \begin{pmatrix} 0 & \sigma_3 \\ 0 & 0 \end{pmatrix}. \tag{4.3}$$

and the transformation matrix become

$$S(\alpha, \beta) = \exp\left(-i\alpha F_1 - i\beta F_2\right) = \begin{pmatrix} I & -\alpha\sigma_1 + \beta\sigma_3 \\ 0 & I \end{pmatrix}. \tag{4.4}$$

We can now write the above operator in terms of creation and annihilation operators. Indeed, if we compare the expressions given in Eq.(2.3), Eq.(2.4) and Eq.(2.6),

$$\hat{S}(\alpha, \beta) = \exp\left(\frac{\alpha}{4}\left\{\left(\hat{a}_1^\dagger - \hat{a}_1\right)^2 - \left(\hat{a}_2^\dagger - \hat{a}_2\right)^2\right\} + \frac{\beta}{2}\left(\hat{a}_1^\dagger - \hat{a}_1\right)\left(\hat{a}_2^\dagger - \hat{a}_2\right)\right). \tag{4.5}$$

The exponent of this expression takes the quadratic form

$$\frac{\alpha}{4}\left(\hat{a}_1^{\dagger 2} + \hat{a}_1^2 - 2\hat{a}_1^\dagger\hat{a}_1\right) - \frac{\alpha}{4}\left(\hat{a}_2^{\dagger 2} + \hat{a}_2^2 - 2\hat{a}_2^\dagger\hat{a}_2\right) + \frac{\beta}{2}\left(\hat{a}_1^\dagger\hat{a}_2^\dagger + \hat{a}_1\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_1 - \hat{a}_1^\dagger\hat{a}_2\right). \tag{4.6}$$

This is normal-ordered, but the shear operator $\hat{S}(\alpha, \beta)$ is not. However, there are theorems in the literature which allow us to write the $\hat{S}$ operator in a normal-ordered form [13]. Indeed, $\hat{S}(\alpha, \beta)$ can be written as

$$\hat{S}(\alpha, \beta) = \lambda :\exp\left\{\xi\left(\hat{a}_1^{\dagger 2} + \hat{a}_1^2 - 2\hat{a}_1^\dagger\hat{a}_1\right) + \eta\left(\hat{a}_2^{\dagger 2} + \hat{a}_2^2 - 2\hat{a}_2^\dagger\hat{a}_2\right)\right.$$

$$\left. + \zeta\left(\hat{a}_1^\dagger\hat{a}_2^\dagger + \hat{a}_1\hat{a}_2 - \hat{a}_1^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_1\right)\right\} :, \tag{4.7}$$

with

$$\lambda = \frac{2}{\sqrt{\alpha^2 + \beta^2 + 4}}, \quad \xi = \frac{\alpha^2 + \beta^2 - 2i\alpha}{2\left(\alpha^2 + \beta^2 + 4\right)}, \quad \eta = \frac{\alpha^2 + \beta^2 + 2i\alpha}{2\left(\alpha^2 + \beta^2 + 4\right)}, \quad \zeta = \frac{-2i\beta}{\alpha^2 + \beta^2 + 4}. \tag{4.8}$$

If this operator is applied to the vacuum state, the annihilation operators deleted. The result is

$$\hat{S}(\alpha, \beta)|0, 0> = \hat{T}(\alpha, \beta)|0, 0>, \tag{4.9}$$

where

$$\hat{T}(\alpha, \beta) = \lambda \exp\left(\xi \hat{a}_1^{\dagger 2} + \eta \hat{a}_2^{\dagger 2} + \zeta \hat{a}_1^{\dagger} \hat{a}_2^{\dagger}\right). \tag{4.10}$$

This operator can now be decomposed into

$$\hat{T}(\alpha, \beta) = \lambda \exp\left(\xi \hat{a}_1^{\dagger 2}\right) \exp\left(\eta \hat{a}_2^{\dagger 2}\right) \exp\left(\zeta \hat{a}_1^{\dagger} \hat{a}_2^{\dagger}\right). \tag{4.11}$$

If this operator is acted on the vacuum state,

$$|\alpha, \beta> = \hat{T}(\alpha, \beta)|0> = \sum_{k,j} C_{kj}|k, j>, \tag{4.12}$$

where k and j are the Fock-space indices for the first and second photons respectively. Using the form of Eq.(4.8), it is possible to calculate the coefficient C_{kj}. The results are

$$C_{2m+1,2n} = C_{2m,2n+1} = 0,$$

$$C_{2m,2n} = \lambda \sqrt{(2m)!(2n)!} \sum_{k=0}^{Min(m,n)} \frac{\zeta^{2k} \xi^{m-k} \eta^{n-k}}{(2k)!(m-k)!(n-k)!},$$

$$C_{2m+1,2n+1} = \lambda \sqrt{(2m+1)!(2n+1)!} \sum_{k=0}^{Min(m,n)} \frac{\zeta^{2k+1} \xi^{m-k} \eta^{n-k}}{(2k+1)!(m-k)!(n-k)!}. \tag{4.13}$$

When $\beta = 0, C_{2m+1,2n+1} = 0$, and the series of Eq.(4.12) becomes a separable expansion.

The distribution $|C_{k,j}|^2$ of course depends on k and j which are the numbers for the first and second photons. These coefficients are normalized. The average value of the number of the first and second photons are

$$<N_1> = \sum_{k=0}^{\infty} k|C_{k,j}|^2, \qquad <N_2> = \sum_{j=0}^{\infty} j|C_{k,j}|^2. \tag{4.14}$$

Similarly,

$$<N_1 N_2> = \sum_{k=0}^{\infty} kj|C_{k,j}|^2, \tag{4.15}$$

and

$$<N_1^2> = \sum_{k=0}^{\infty} k^2|C_{k,j}|^2, \qquad <N_2^2> = \sum_{j=0}^{\infty} j^2|C_{k,j}|^2. \tag{4.16}$$

The computation of these numbers in terms of the coefficients given in Eq.(4.8) is possible. On the other hand, it is not clear whether this computation will lead to a closed analytical form for each of the above quantities. The method of Wigner function will make these calculations relatively simple [3,10].

In this report, we have given only a sample calculation. We have not made any attempt to derive the numbers which can be measured in laboratories. This can be done when there are more concrete evidence for sheared states of photons or phonons [8,9].

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